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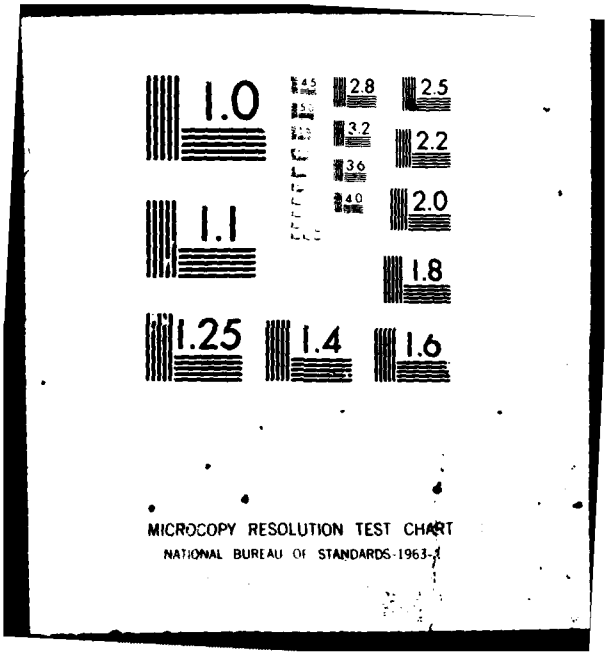
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by

(10) Henry W. Block and Mei-Ling Ting

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Department of Mathematics and Statistics

University of Pittsburgh

Pittsburgh, PA 15260

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Some Concepts of Multivariate Dependence [1]

by

Henry W. Block and Mei-Ling Ting

University of Pittsburgh

Abstract

Several types of positive dependence are shown to be equivalent to the concept of a distribution with a density which is TP_2 in pairs. Among these is the concept of m*-positive dependence of Alam and Wallenius Using this result, all relationships among many of the most important concepts of positive dependence are determined. Furthermore, an application of the equivalences of these types of positive dependence yields a result of Ahmed, Langberg, Leon and Proschan.

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1. Introduction

In a wide variety of situations, one is interested in the probability that an n-variate random vector $\underset{\sim}{X} = (X_1,\ldots X_n)$ takes certain values. If the vector does not have independent components, these probabilities can be quite difficult to obtain. However, in many applications it is often the case that

$$P\{X_1 \leq x_1,\ldots,X_n \leq x_n\} \geq \prod_{i=1}^{n} P\{X_i \leq x_i\} \tag{1.1}$$

or

$$P\{X_1 > x_1,\ldots,X_n > x_n\} \geq \prod_{i=1}^{n} P\{X_i > x_i\}. \tag{1.2}$$

That is, the joint probabilities are higher than if the random variables were independent and consequently the joint probabilities can be estimated from below through only the knowledge of the marginal distributions. Random vectors which satisfy these basic inequalities are said to be positively orthant dependent. These concepts have recently been studied by Ahmed, Langberg, Leon and Proschan (1978) (henceforth ALLP (1978)) and by many other authors cited in that paper.

A stronger type of positive dependence is exemplified by the situation where

$$P\{X_i > x_i \mid X_j > a_j,\ j \neq i\} \text{ increases in } a_j. \tag{1.3}$$

This is a special case of a concept studied by Harris (1970) and also by Alam and Wallenius (1976). This condition arises in situations where X_i can be thought of as an unobservable performance variable and the X_j are observable test variables. The condition can be interpreted as meaning that the probability of the performance variable increases if the test variable requirements are increased.

One of the aims of the paper of Alam and Wallenius (1976) as well as many other authors (e.g., see Chapter 5 of Barlow and Proschan) is to obtain simple, checkable, conditions under which this latter type of dependence as well as (1.1) and (1.2) hold. Although in the bivariate case, these results are known, in the general multivariate case there have only been fragmentary results. In particular, Alam and Wallenius (1976) show that a certain condition, called m*-dependence, implies (1.3), while Barlow and Proschan (1975) summarize some concepts which imply (1.1) and (1.2). In particular, these latter authors show that if a distribution has a density which is TP_2 in pairs, then the distribution satisfies several other concepts of positive dependence.

In this paper we show in the Theorem of Section 2, that the Alam and Wallenius concept of m*-dependence is actually the same as the concept of TP_2 in pairs, a property satisfied by a wide variety of multivariate distributions. Hence, as a corollary to this Theorem we obtain that all of these distributions have the dependence properties studied by these authors. Furthermore, we show that several other concepts are equivalent to the concept of TP_2 in pairs. As another corollary we obtain a general result of ALLP (1978) on positive dependence, a result which required a much more involved proof originally.

In Section 3, as a consequence of Section 2, we are able to completely determine all relationships among the various concepts of positive dependence studied by the authors mentioned above.

2. Definitions and Main Result

Let $\underset{\sim}{X} = (X_1,\ldots,X_n)$ be an n-variate random vector with density $f(\underset{\sim}{x}) = f(x_1,\ldots,x_n)$. We say $\underset{\sim}{X}$ or f is $\underline{TP_2}$ $\underline{\text{in pairs}}$ if for any $i \neq j$ and for $x_i < x_i'$, $x_j < x_j'$

$$f(x_i,x_j,\underset{\sim}{x}^{(i,j)})f(x_i',x_j',\underset{\sim}{x}^{(i,j)}) \leq f(x_i',x_j,\underset{\sim}{x}^{(i,j)})f(x_i,x_j',\underset{\sim}{x}^{(i,j)})$$

for all $\underset{\sim}{x}^{(i,j)} = (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{j-1},x_{j+1},\ldots,x_n)$. Multivariate distributions which satisfy this can be found in the examples and problems of Chapter 5 of Barlow and Proschan (1975) and also in Karlin (1968), p. 19.

Throughout the paper we use the term "increasing" to mean "non-decreasing".

Alam and Wallenius (1976) define the components of the random vector $\underset{\sim}{X} = (X_1,X_2,\ldots,X_n)$ to be $\underline{\text{m*-positively dependent}}$, if for $i = 1,\ldots,n$ the conditional distribution of X_i given $\underset{\sim}{X}^{(i)} = (X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_n)$ is positively likelihood ratio dependent on $\underset{\sim}{X}^{(i)}$, i.e. if the conditional density $g(x_i|\underset{\sim}{x}^{(i)})$ exists such that, whenever $x_i' \geq x_i$ and $\underset{\sim}{x}'^{(i)} \geq \underset{\sim}{x}^{(i)}$

$$g(x_i'|\underset{\sim}{X}^{(i)} = \underset{\sim}{x}'^{(i)})g(x_i|\underset{\sim}{X}^{(i)} = \underset{\sim}{x}^{(i)}) \geq g(x_i'|\underset{\sim}{X}^{(i)} = \underset{\sim}{x}^{(i)})g(x_i|\underset{\sim}{X}^{(i)} = \underset{\sim}{x}'^{(i)}),$$

for $i = 1,2,\ldots,n$.

Ahmed, Langberg, Leon and Proschan (1978) define $\underset{\sim}{X}$ to be $\underline{\text{totally positive by deletion (TPD)}}$ if $\underset{\sim}{X}$ has a density f satisfying

$$\begin{vmatrix} f(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n) & f(x_1,\ldots,x_{i-1},x_i',x_{i+1},\ldots,x_n) \\ f(x_1',\ldots,x_{i-1}',x_i,x_{i+1}',\ldots,x_n') & f(x_1',\ldots,x_{i-1}',x_i',x_{i+1}',\ldots,x_n') \end{vmatrix} \geq 0$$

for all $x_i \leq x_i'$ $i = 1,2,\ldots,n$.

Consider the notations $\underset{\sim}{x} \vee \underset{\sim}{y} = (x_1 \vee y_1,\ldots,x_n \vee y_n)$ and $\underset{\sim}{x} \wedge \underset{\sim}{y} = (x_1 \wedge y_1,\ldots,x_n \wedge y_n)$, where $x_k \vee y_k = \max\{x_k, y_k\}$, $x_k \wedge y_k = \min\{x_k, y_k\}$ and let $(x_1,\ldots,x_n) \leq (y_1,\ldots,y_n)$ iff $x_k \leq y_k$ for all $k = 1,\ldots,n$.

Kemperman (1977) shows, in a more general setting than we consider here, that if $X_1,\ldots,X_n$ have the joint density $f(\underset{\sim}{x})$ where $f(\underset{\sim}{x}) > 0$ for $\underset{\sim}{x} \in \Omega = \prod_{i=1}^{n} \Omega_i$, $\Omega_i \subset R^1$ and if $f(\underset{\sim}{x})$ is TP_2 in pairs, then $f(\underset{\sim}{x})\, f(\underset{\sim}{y}) \leq f(\underset{\sim}{x} \wedge \underset{\sim}{y})\, f(\underset{\sim}{x} \vee \underset{\sim}{y})$ for all $\underset{\sim}{x}, \underset{\sim}{y} \in \Omega$.

This last result is important in the following theorem in which it is shown that these four concepts are equivalent.

Theorem 1. Let $X_1,\ldots,X_n$ have joint density $f(\underset{\sim}{x})$, where $f(\underset{\sim}{x}) > 0$ for all $\underset{\sim}{x} \in \Omega = \prod_{i=1}^{n} \Omega_i$, $\Omega_i \subset R^1$, then the following four statements are equivalent;

(1) $X_1, X_2, \ldots, X_n$ are m*-positively dependent

(2) $X_1, X_2, \ldots, X_n$ are TPD

(3) $f(x_1, x_2, \ldots, x_n)$ is TP_2 in pairs

(4) $f(\underset{\sim}{x})f(\underset{\sim}{y}) \leq f(\underset{\sim}{x} \wedge \underset{\sim}{y})f(\underset{\sim}{x} \vee \underset{\sim}{y})$ $\quad \underset{\sim}{x}, \underset{\sim}{y} \in \Omega \subset R^n$.

Proof:

(1) $\Rightarrow$ (2) This result is obvious.

(2) $\Rightarrow$ (3) In the definition of TPD take any $x_i \leq x_i'$, $x_j \leq x_j'$ for $j \neq i$ and take $x_k = x_k'$ for $k = 1,2,\ldots,n$ with $k \neq j$, $k \neq i$.

(3) $\Rightarrow$ (4) (Kemperman 1977)

By permuting the indices if necessary, we may consider $\underset{\sim}{x},\underset{\sim}{y} \in R^n$ such that $x_k \geq y_k$ for $k = 1,\ldots,r$, and $x_k \leq y_k$ for $k = r+1,\ldots,n$. Let $s = n-r$. Now for $0 \leq i \leq r$, $0 \leq j \leq s$ define:

$$\underset{\sim}{z}^{i,j} = (x_1 \vee y_1,\ldots,x_i \vee y_i, x_{i+1} \wedge y_{i+1},\ldots,x_r \wedge y_r,$$
$$x_{r+1} \vee y_{r+1},\ldots,x_{r+j} \vee y_{r+j}, x_{r+j+1} \wedge y_{r+j+1},\ldots,x_n \wedge y_n)$$

then $\underset{\sim}{z}^{r,0} = \underset{\sim}{x}$, $\underset{\sim}{z}^{0,s} = \underset{\sim}{y}$, $\underset{\sim}{z}^{0,0} = \underset{\sim}{x} \wedge \underset{\sim}{y}$, $\underset{\sim}{z}^{r,s} = \underset{\sim}{x} \vee \underset{\sim}{y}$.

Since $\underset{\sim}{z}^{i+1,j}$ and $\underset{\sim}{z}^{i,j+1}$ differ only at the $(i+1)^{th}$ and the $(r+j+1)^{th}$ coordinates, $\underset{\sim}{z}^{i+1,j} \wedge \underset{\sim}{z}^{i,j+1} = \underset{\sim}{z}^{i,j}$, $\underset{\sim}{z}^{i+1,j} \vee \underset{\sim}{z}^{i,j+1} = \underset{\sim}{z}^{i+1,j+1}$, and since the density is TP_2 in pairs

$$1 \geq \prod_{i=0}^{r-1}\prod_{j=0}^{s-1} \frac{f(\underset{\sim}{z}^{i+1,j})f(\underset{\sim}{z}^{i,j+1})}{f(\underset{\sim}{z}^{i,j})f(\underset{\sim}{z}^{i+1,j+1})} = \frac{f(\underset{\sim}{z}^{0,s})f(\underset{\sim}{z}^{r,0})}{f(\underset{\sim}{z}^{0,0})f(\underset{\sim}{z}^{r,s})} = \frac{f(\underset{\sim}{x})f(\underset{\sim}{y})}{f(\underset{\sim}{x} \wedge \underset{\sim}{y})f(\underset{\sim}{x} \vee \underset{\sim}{y})}$$

(4) $\Rightarrow$ (1) Take $\underset{\sim}{x} = (x_i,\underset{\sim}{x}'^{(i)})$, $\underset{\sim}{y} = (x_i',\underset{\sim}{x}^{(i)})$ where $x_k \leq x_k'$, then $\underset{\sim}{x} \wedge \underset{\sim}{y} = (x_i,\underset{\sim}{x}^{(i)})$, $\underset{\sim}{x} \vee \underset{\sim}{y} = (x_i',\underset{\sim}{x}'^{(i)})$ and $f(\underset{\sim}{x})f(\underset{\sim}{y}) \geq f(\underset{\sim}{x} \wedge \underset{\sim}{y})f(\underset{\sim}{x} \vee \underset{\sim}{y})$. (1) follows just by dividing through on both sides by appropriate marginal densities which must be positive on the appropriate projections of Ω.

Note: As amentioned in Kemperman it should be noticed that (3) does not imply (2) without the assumption that $f(\underset{\sim}{x}) > 0$ on Ω of the form in the theorem. The simple counterexample given is

$f(x_1,x_2,x_3) = \frac{1}{2}$ if $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 1$, $1 < x_3 \leq 2$,
$\frac{1}{2}$ if $1 < x_1 \leq 2$, $1 < x_2 \leq 2$, $0 \leq x_3 \leq 1$,
0 otherwise.

Several corollaries to the above result can now be stated. The first is a new result and allows us to relate the various multivariate concepts of positive dependence. The second corollary is a slightly more general version of the interesting result of ALLP (1978) and this follows immediately from the first corollary.

Corollary 1: Let $(X_1,\ldots,X_n)$ have a density $f(x_1,\ldots,x_n)$ which is TP_2 in pairs on the set Ω in the Theorem.

$$E(\psi(X_1,\ldots,X_n)|X_i > a_i,\ i = 1,\ldots,n) \text{ is increasing in } a_1,\ldots,a_n.$$

First Proof: This follows from the previous theorem and Theorem 3.3 of Alam and Wallenius (1976) which remains true under the assumption that $f(\underset{\sim}{X}) > 0$ on $\Omega = \prod_{i=1}^{n}\Omega_i$, $\Omega_i \subset R^1$

Second Proof: We also prove this result by giving a streamlined version of the proof of Theorem 3.3. It is sufficient to prove the result by checking the monotonicity only for a_1. We write

$$E(\psi(\underset{\sim}{X})|X_i > a_i,\ i = 1,\ldots,n) \tag{2.1}$$

$$= E(\ldots E(\ldots E(\psi(\underset{\sim}{X})|X_1,\ldots,X_n)\ldots|X_1,\ldots,X_{i-1},X_i>a_i,\ldots,X_n>a_n)\ldots|X_i>a_i,\ i=1,\ldots,n).$$

Now the TP_2 condition implies that X_i is stochastically increasing in $X_j = x_j$, $j = 1,\ldots,i-1$ given $X_k > a_k$, $k = i,\ldots,n$. Thus, the quantity $E(\ldots E(\psi(\underset{\sim}{X})|\ldots|X_1 = x_1,\ldots,X_{i-1} = x_{i-1},X_i > a_i,\ldots,X_n > a_n)$ is increasing in x_j, $j = 1,\ldots,i-1$. Iterating gives that the quantity inside of the first expectation of (2.1) is increasing in x_1. Letting this quantity be $\phi(X_1)$ yields that

$$E(\psi(\underset{\sim}{X})|X_i > a_i,\ i = 1,\ldots,n) = E(\phi(X_1)|X_i > a_i,\ i = 1,\ldots,n) \text{ increases in } a_1$$

since the density with which the last expectation is taken is TP_2 (i.e. has monotone likelihood ratio) in a_1 and x_1.

Corollary 2: Let $X_1,\ldots,X_n$, given λ, a scalar random variable, be independent with the densities $f_i(x_i,\lambda)$ which are each TP_2 in x_i and λ for $i = 1,\ldots,n$. Then $X_1,\ldots,X_n$ satisfy

$$P\{X_i > x_i \mid X_j > x_j,\ j \in I\} \text{ are increasing in } x_j$$

where $I \subset \{1,2,\ldots,i-1,i+1,\ldots,n\}$. In particular, $X_1,\ldots,X_n$ are what ALLP (1978) call right tail increasing in sequence (see the next section).

Proof: By the basic composition formula (Karlin (1968), p. 16) it follows that the density $f(x_1,\ldots,x_n) = \int_{i=1}^{n} f_i(x_i,\lambda)g(\lambda)d\lambda$ is TP_2 in pairs. By Corollary 1 the result follows by appropriate choice of ψ and allowing certain $a_i \to -\infty$.

3. Relations

The implications among the various notions of bivariate dependence are summarized in Figure 5.4.1, p. 146 of Barlow and Proschan (1975). In the same book are introduced the multivariate ($m > 3$) concepts of TP_2 in pairs, conditionally increasing in sequence (CIS), and association. Brindley and Thompson (1972) have discussed the concept of right corner set increasing (RCSI). Other concepts discussed by ALLP (1978) are given below. In this section we completely determine the relationships among all of these concepts.

Definition 1: A sequence of random variables $X_1,\ldots,X_n$ is said to be right tail increasing in sequence (RTIS) if $P(X_{i+1} > x_{i+1} \mid \bigcap_{j=1}^{i} \{X_j > x_j\})$ is increasing in $x_1,\ldots x_i$ for $i = 1,\ldots,n-1$.

Definition 2: The random variables $X_1,\ldots,X_n$ are mutually positive orthant dependent (POD) if

$$P(\bigcap_{i=1}^{n} \{X_i > x_i\}) \geq \prod_{i=1}^{n} P(X_i > x_i) \quad \text{for all} \quad x_1,\ldots,x_n.$$

Note that for $n = 2$, POD coincides with the concept of positive quadrant dependence (PQD) studied in Lehmann (1966).

Alam and Wallenius (1976) introduce the concept of s*-positive dependence which in terms of the above is equivalent to every sequence being CIS. These authors also show that m*-positively dependent implies RCSI. Hence, by Corollary 1 we have TP_2 in pairs implies RCSI. Ahmed, Langberg, Leon and Proschan show that

$$\text{RCSI} \Rightarrow \text{RTIS} \not\Rightarrow \text{POD}.$$

The first implication is also shown in Brindley and Thompson (1972).

Barlow and Proschan (1975) show that TP_2 in pairs implies conditional increasing in sequence (CIS). And the result that CIS implies association is given by Esary and Proschan (1968). The proof that association satisfies the condition of POD is in Esary, Proschan and Walkup (1967).

Since in the bivariate case RTIS implies association, (see Esary and Proschan (1972)), it seems reasonable to conjecture that

RTIS implies association in the multivariate case. However, this is not true; as will be seen by a counterexample. Brindley and Thompson (1972) show that RCSI does not imply association.

Association does not imply RTIS (see Esary and Proschan (1972)). Hence, association does not imply RCSI neither.

We also give examples to show that RTIS neither implies nor is implied by CIS. These results were noted by ALLP (1978), but their example needs to be modified.

For RCSI and CIS, there is no implication between them even in bivariate case (see Barlow and Proschan (1975)).

We summarize the implications among the notions of multivariate dependence as in the following diagram, where $I(X_1,\ldots,X_n)$ means $X_1,\ldots,X_n$ are independent.

$$I(X_1,\ldots,X_n) \Rightarrow TP_2 \text{ in pairs} \Rightarrow \begin{matrix} \text{RCSI} & & \text{RTIS} \\ & & \searrow \\ & & \text{POD} \\ \text{CIS} & & \text{association} \end{matrix}$$

Any other implication among CIS, RCSI, RTIS and association does not hold. In the above it is understood that the implications yielding RTIS and CIS hold for every sequence, but the implications following from these concepts require the condition for only one sequence.

Example 1: (RTIS $\not\Rightarrow$ Association)

Let X_1, X_2 and X_3 be binary random variables distributed according to

$$P(X_1 = 0, \quad X_2 = 0, \quad X_3 = 1) = 0.2$$

$$P(X_1 = 0, \quad X_2 = 1, \quad X_3 = 0) = 0.1$$

$$P(X_1 = 1, \quad X_2 = 0, \quad X_3 = 0) = 0.1$$

$$P(X_1 = 1, \quad X_2 = 1, \quad X_3 = 0) = 0.1$$

$$P(X_1 = 1, \quad X_2 = 1, \quad X_3 = 1) = 0.5 \ .$$

It is easy to check that X_1, X_2 and X_3 are RTIS. Let I_{U_1}, I_{U_2} be the indicator functions of the sets U_1, U_2 and where

$$U_1 = \{(0,0,1), (0,1,0), (1,0,1), (0,1,1), (1,1,0), (1,1,1)\}$$

and

$$U_2 = \{(0,1,0), (1,0,0), (1,0,1), (0,1,1), (1,1,0), (1,1,1)\}.$$

Then I_{U_1}, I_{U_2} are binary increasing functions. But $\mathrm{Cov}[I_{U_1}(X_1,X_2,X_3), \ I_{U_2}(X_1,X_2,X_3)] = P[(X_1,X_2,X_3) \in U_1 \cap U_2] - P[(X_1,X_2,X_3) \in U_1] \, P[(X_1,X_2,X_3) \in U_2] = 0.7 - 0.72 < 0$. Thus, X_1, X_2 and X_3 are not associated.

Example 2: (CIS $\not\Rightarrow$ RTIS)

Let X_3, given $(X_1,X_2) = (x_1,x_2)$ be distributed according to the normal $N(x_1+x_2,1)$ and let (X_1,X_2) be jointly distributed according to

X_1 \ X_2	-1	0	2
-1	0.3	0	0.1
0	0.2	0	0.1
1	0.1	0.1	0.1 .

Then X_1, X_2 and X_3 are CIS, but

$$P(X_3 > 2 \mid X_1 > -1, X_2 > -1) = 0.2774$$

$$> P(X_3 > 2 \mid X_1 > 0, X_2 > -1) = 0.1587$$

i.e. X_1, X_2 and X_3 are not RTIS.

Example 3: (RTIS $\not\Rightarrow$ CIS)

Let X_3, Given $(X_1, X_2) = (x_1, x_2)$ be defined as in Example 2, and let X_1, X_2 be distributed such that

X_1 \ X_2	-1	0	1
-1	0.05	0.25	0
0	0.1	0	0.2
1	0	0.1	0.3

then it can be checked that X_1, X_2 and X_3 are RTIS but

$$P(X_2 > -1 \mid X_1 = -1) = 0.25/0.3$$

$$P(X_2 > -1 \mid X_1 = 0) = 0.2/0.3 < 0.25/0.3$$

therefore X_1, X_2 and X_3 are not CIS.

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